Discovering 5-Valent Semi-Symmetric Graphs

Berkeley Churchill

NSF REU in Mathematics
Northern Arizona University
Flagstaff, AZ 86011

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Groups and Graphs

- Graphs are taken to be simple (no loops, multiloops), undirected and unweighted.
- Let $\Gamma_1, \Gamma_2$ be graphs. $\phi : \Gamma_1 \to \Gamma_2$ is an isomorphism of graphs if $\phi : V(\Gamma_1) \to V(\Gamma_2)$ and $\phi : E(\Gamma_1) \to E(\Gamma_2)$ are bijections and adjacency between edges and vertices are preserved under $\phi$. In some sense this means $\Gamma_1$ and $\Gamma_2$ are “the same”.
- The isomorphisms from $\Gamma_1$ to itself form the automorphism group, denoted $\text{Aut}(\Gamma_1)$. These automorphisms are called symmetries.
- Alternatively, a symmetry is a permutation of the vertices that preserves edge-adjacency.
- If $v \in V(\Gamma)$ and $\phi$ is a symmetry of $\Gamma$ then $v$ and $\phi(v)$ have the same local properties.
Group Actions

- Suppose $G$ is a group and $S$ is a set. $\text{Sym}(S)$ is the group of all bijections on $S$. A group action of $G$ on $S$ is a group homomorphism $\phi: G \rightarrow \text{Sym}(S)$.
- If $x \in S$ and $g \in G$ then $xg$ denotes $\phi(x)(g)$.
- $\text{Orb}_G(x) = \{xg | g \in G\}$.
- $\text{Stab}_G(x) = \{g | xg = x\}$.
- $\phi$ is transitive if for all $x \in S$, $\text{Orb}_G(x) = S$.

For a graph $\Gamma$, $\text{Aut}(\Gamma)$ acts on both $V(\Gamma)$ and $E(\Gamma)$. We talk about edge stabilizers and vertex stabilizers to mean the automorphisms fixing a particular edge or vertex.

For vertices (or edges) in the same orbit, the vertex (edge) stabilizers are all the same, along with other local properties.
Semi-Symmetric Graphs

- $\Gamma$ is edge-transitive if $\text{Aut}(\Gamma)$ acts transitively on $E(\Gamma)$. This means that every edge has the same local properties.
- $\Gamma$ is vertex-transitive if $\text{Aut}(\Gamma)$ acts transitively on $V(\Gamma)$. Again, this means that every vertex has the same local properties.
- These two are independent; neither implies the other.
- Semi-Symmetric graphs are graphs which are edge-transitive, not vertex-transitive and regular.
- All edge-transitive graphs fall into one of the following three categories: symmetric, strongly bi-transitive, $\frac{1}{2}$-arc-transitive.
- Semi-symmetric graphs are strongly-bi-transitive graphs that are regular.

<table>
<thead>
<tr>
<th>Dart-Transitive</th>
<th>Vertex-Transitive</th>
<th>Not Vertex-Transitive</th>
</tr>
</thead>
<tbody>
<tr>
<td>Symmetric</td>
<td>$\frac{1}{2}$-Arc transitive</td>
<td>Impossible</td>
</tr>
<tr>
<td>Strongly Bi-Transitive</td>
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Properties of Semi-Symmetric Graphs

- Edge-Transitive but not Vertex-Transitive and Regular
- Bi-partite (and therefore \textit{bi-transitive})
- there is no symmetry that interchanges a white vertex with a black vertex
- The orbit of a vertex includes every vertex of the same color.
  - white vertices “look” the same and black vertices “look” the same.
  - preserved properties include stabilizers and distances to vertices of a given color

\textbf{Figure:} Folkman’s graph, the smallest semi-symmetric graph. [2]
Problem Statement

- What is the smallest 5-valent semi-symmetric graph?
  - typically proving this is hard, and is done either by enumeration or long combinatorial arguments
  - algorithms to brute-force enumerate edge-transitive graphs are too expensive to get past 30 vertices
  - Conder et al. is an exception, where they use powerful results from Goldschmidt to classify graphs [1].

- How can we construct 5-valent semi-symmetric graphs?
  - are there easy constructions?
  - can we find an infinite family?
Previous Results

- The smallest semi-symmetric graph is Folkman’s graph on 20 vertices and 40 edges.
- The smallest 3-valent semi-symmetric graph is the Gray graph on 54 vertices.
- A semi-symmetric graph must have \( n \) vertices where \( n \) is even and not \( 2p \) or \( 2p^2 \) for any prime \( p \). [2]

Figure: Folkman’s graph, the smallest semi-symmetric graph. [2]
Bi-Coset Construction

Let $G$ be a group and let $H, K$ be subgroups. Construct a graph $\Gamma = \text{bcc}(G; H, K)$ with

$$V(\Gamma) = G/H \cup G/K$$
$$E(\Gamma) = \{(Hg, Kg) | g \in G\}$$

- $Hg_1$ is adjacent to $Kg_2$ if and only if $Hg_1 \cap Kg_2 \neq \emptyset$.
- $d$-valent $\iff [H : H \cap K] = [K : H \cap K] = d$.
- connected $\iff \langle H, K \rangle = G$.
- always edge-transitive, bi-partite (bi-transitive).
- all bi-transitive graphs come from this construction: pick $u, v \in V(\Gamma)$ adjacent, let $H = \text{Stab}_G(u), K = \text{Stab}_G(v)$. Then $\text{bcc}(G; H, K) \cong \Gamma$.
- For all $g \in G$, $\text{bcc}(G; H, K) \cong \text{bcc}(G; g^{-1}Hg, g^{-1}Kg)$. 
Bi-Coset Construction Searches

How to find \( d \)-valent semi-symmetric graphs:

1. Pick a finite group \( G \) from a database.
2. For each \( H \leq G \) with \( d \mid \#H \), consider a representative \( K \) of every conjugacy class of subgroups that can satisfy \([H : H \cap K] = [K : H \cap K] = d\).
3. Compute \( \Gamma \cong \text{bcc}(G; H, K) \).
4. Determine if \( \Gamma \) is vertex-transitive.

Searching all finite groups of size less than 1200, I have found three 5-valent semi-symmetric graphs. Only one of these, with 250 vertices, was previously discovered by Lazebnik and Viglione [3].

Question: what are the graphs we have found?
What has been found?

- If $\Gamma$ is bi-transitive, degree $d$ and $\text{Aut}^+(\Gamma)$ has a subgroup of size $n \leq 1200$ transitive on the edges of $\Gamma$ then $\Gamma$ was found by the search.

- In the case where $H \leq \text{Aut}(\Gamma)$ is edge-transitive and $|H| = |E(\Gamma)|$ I have found all semi-symmetric 5-valent graphs with less than 1200 edges.

- When $H \leq \text{Aut}(\Gamma)$ acts on the edges this way, it acts *regularly*. Namely, for $e_1, e_2 \in E(\Gamma)$ there exists exactly one $h \in H$ so that $e_1 h = e_2$. Equivalently, the dart-stabilizers in $H$ are trivial. I call a graph with such an action *edge-regular*.

- Therefore, every edge-regular semi-symmetric graph with less than 1200 edges has been found.

Question: how can I classify which graphs are edge-regular?  
Better Question: how can I classify which graphs are not edge-regular?
Cayley Graphs

Let $G$ be a group and $S \subseteq G$. Define $\Gamma = \text{Cay}(G, S)$ to be the (undirected!) graph with $V(\Gamma) = G$ and $E(\Gamma) = \{\{g, sg\} | g \in G\}$.

Figure: $\text{Cay}(D_4, \{a, b\})$ where $a$ (red) is rotation and $b$ (blue) is reflection.

- $G$ acts regularly on the vertices of $\text{Cay}(G, S)$. 
Line Graphs

Let $\Delta$ be a graph. Define $\Gamma = L(\Delta)$ so that $V(\Gamma) = E(\Delta)$ and two vertices of $\Gamma$ are adjacent when the corresponding edges are adjacent.

$L(\Delta)$ usually has more edges than $\Delta$ has vertices.

$\text{Aut}(L(K_4)) \cong S_4 \times S_2$

$\text{Aut}(K_4) \cong S_4$

$\text{Aut}(L(\Delta)) \neq \text{Aut}(\Delta)$ in general.
Line graphs of Edge-Regular Graphs are Cayley Graphs

Motivation: Cayley graphs are vertex-regular, and line graphs “switch” edges and vertices!

Lemma

If $G$ is a group with $H, K \leq G$, $H \cap K = 1$ and $\langle H, K \rangle = G$ then
$L(bcc(G; H, K)) \cong \text{Cay}(G, H \cup K - \{1\})$.

Proof.

Explicitly construct the vertex and edge set of $L(bcc(G; H, K))$. They match $\text{Cay}(G, H \cup K - \{1\})$ exactly.

Theorem

A connected bi-transitive graph $\Delta$ is edge-regular if and only if there exists a group $G$ and a subset $S \subset G$ such that $L(\Delta) \cong \text{Cay}(G, S)$. 
Proof of Theorem

**Theorem**

A connected bi-transitive graph $\Delta$ is edge-regular if and only if there exists a group $G$ and a subset $S \subseteq G$ such that $L(\Delta) \cong \text{Cay}(G, S)$.

**Proof.**

($\Rightarrow$). Suppose $G \leq \text{Aut}(\Delta)$ acts regularly on the edges of $\Delta$. Pick $H$ and $K$ to be stabilizers of an adjacent white and black vertex in $G$, respectively. $H \cap K$ is a dart-stabilizer, so $H \cap K = 1$. $\Delta \cong \text{bcc}(G; H, K)$. The lemma establishes that $L(\Delta) \cong \text{Cay}(G, H \cup K - \{1\})$.

($\Leftarrow$). Outline: Let $\Gamma = L(\Delta)$. For $\Delta$ bi-partite, $\text{Aut}(\Gamma)$ acts on $E(\Delta)$ in the same way that $\text{Aut}(\Gamma)$ acts on $V(\Gamma)$ (demonstrated on the next slide). If $\Gamma = \text{Cay}(G, S)$, then $G \leq \text{Aut}(\Gamma)$ acts regularly on the vertices of $\Gamma$, and therefore $G$ acts regularly on the edges of $\Delta$. □
Lemma

If $\Delta$ is a bi-partite graph, $\Gamma = L(\Delta)$ and $G \leq \text{Aut}(\Gamma)$, then $G$ acts on $\Delta$ as a subgroup of $\text{Aut}(\Delta)$.

Proof.

(Sketch) The edges of $\Gamma$ are colored white and black from the vertices of $\Delta$. Let $K$ be a clique in $\Gamma$ with $|V(K)| \geq 3$. Suppose let $K'$ be an induced subgraph with 3 vertices. By pigeonhole, two edges must be the same color, say red. Then there are two ways $K'$ could be colored:

$$K_1 \Rightarrow L^{-1}(K_1) \quad \quad K_2 \Rightarrow L^{-1}(K_2)$$

$K' = K_2$ is a contradiction; the coloring of $\Delta$ must be violated. Therefore, all cliques are of a single color, and maximal ones correspond to a single vertex in $\Delta$. $G$ permutes maximal cliques preserving vertex adjacencies, so $G$ acts on the vertices of $\Delta$ preserving edge-adjacency.
Worthiness

Lemma

For any prime \( p \), every connected, unworthy, bi-partite, edge-transitive graph with valence \( p \) is isomorphic to \( K_{p,p} \).

Proof.

Suppose \( u_1, \ldots, u_s \) is a maximal set of white vertices that have the same neighbours. By edge-transitivity, all white vertices are partitioned into sets of size \( s > 1 \) that have the same neighbours. If \( v \) is black then its \( p \) neighbours are partitioned into sets of size \( s \) so \( s|p \Rightarrow s = p \). The \( p \) neighbours of \( u_1, \ldots, u_p \) will be black vertices \( v_1, \ldots, v_p \). In turn, their neighbours are exactly \( u_1, \ldots, u_p \). These form a connected component isomorphic to \( K_{p,p} \).

Corollary

Every 5-valent semi-symmetric graph is worthy.
Summary of Results

- Line graphs of edge-regular bi-transitive graphs are Cayley.
  - I have enumerated these graphs through 1200 edges.
  - This has led to conjectures to generalize Marušič's work [4].

- 5-valent semi-symmetric graphs are worthy.

- A candidate for the smallest 5-valent semi-symmetric graph which is minimal.

- An improved census webpage which now includes several 5-valent bi-transitive graphs.
Next Steps

- Find infinite families of 5-valent semi-symmetric graphs.
  - Generalizing voltage graphs for the 5-valent semi-symmetric graphs found may be useful.
  - It may be possible to generalize some 3-valent families such as Marušič's.

- Develop new search techniques to establish whether the 5-valent semi-symmetric graphs of 120 vertices is indeed minimal.

Figure: A 5-valent semi-symmetric graph with 120 vertices.
References

Marston Conder, Aleksander Malnic, Dragan Marusic, and Primz Potocnik.

Jon Folkman.

Felix Lazebnik and Raymond Viglione.

Dragan Marušič.
Questions?

Berkeley Churchill <berkeley@berkeleychurchill.com>

You can find copies of these slides and a link to the mini-census at http://www.berkeleychurchill.com/research