Discovering 5-Valent Semi-Symmetric Graphs

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Groups and Graphs

- Graphs are taken to be simple (no loops, multiloops), undirected and unweighted.
- Let Γ_1, Γ_2 be graphs. $\phi : \Gamma_1 \rightarrow \Gamma_2$ is an *isomorphism of graphs* if $\phi : V(\Gamma_1) \rightarrow V(\Gamma_2)$ and $\phi : E(\Gamma_1) \rightarrow E(\Gamma_2)$ are bijections and adjacency between edges and vertices are preserved under ϕ. In some sense this means Γ_1 and Γ_2 are "the same".
- The isomorphisms from Γ_1 to itself form the *automorphism group*, denoted $\mathrm{Aut}(\Gamma_1)$. These automorphisms are called *symmetries*.
- Alternatively, a symmetry is a permutation of the vertices that preserves edge-adjacency.
- If $v \in V(\Gamma)$ and ϕ is a symmetry of Γ then v and $\phi(v)$ have the same local properties.

Group Actions

- Suppose G is a group and S is a set. $\mathrm{Sym}(S)$ is the group of all bijections on S. A *group action of G on S* is a group homomorphism $\phi : G \to \mathrm{Sym}(S)$.
- If $x \in S$ and $g \in G$ then xg denotes $\phi(x)(g)$.
- $\mathrm{Orb}_G(x) = \{xg | g \in G\}$.
- $\mathrm{Stab}_G(x) = \{g | xg = x\}$.
- ϕ is transitive if for all $x \in S$, $\mathrm{Orb}_G(x) = S$.

For a graph Γ, $\mathrm{Aut}(\Gamma)$ acts on both $V(\Gamma)$ and $E(\Gamma)$. We talk about *edge stabilizers* and *vertex stabilizers* to mean the automorphisms fixing a particular edge or vertex.

For vertices (or edges) in the same orbit, the vertex (edge) stabilizers are all the same, along with other local properties.

Semi-Symmetric Graphs

- Γ is edge-transitive if $\mathrm{Aut}(\Gamma)$ acts transitively on $E(\Gamma)$. This means that every edge has the same local properties.
- Γ is vertex-transitive if $\mathrm{Aut}(\Gamma)$ acts transitively on $V(\Gamma)$. Again, this means that every vertex has the same local properties.
- These two are independent; neither implies the other.
- Semi-Symmetric graphs are graphs which are edge-transitive, not vertex-transitive and regular.
- All edge-transitive graphs fall into one of the following three categories: symmetric, strongly bi-transitive, $\frac{1}{2}$-arc-transitive.
- semi-symmetric graphs are strongly-bitransitive graphs that are regular.

	Vertex-Transitive	Not Vertex-Transitive
Dart-Transitive	Symmetric	Impossible
Not Dart-Transitive	$\frac{1}{2}$-Arc transitive	Strongly Bi-Transitive

Properties of Semi-Symmetric Graphs

- Edge-Transitive but not Vertex-Transitive and Regular
- Bi-partite (and therefore *bi-transitive*)
- there is no symmetry that interchanges a white vertex with a black vertex
- The orbit of a vertex includes every vertex of the same color.
  - white vertices “look” the same and black vertices “look” the same.
  - preserved properties include stabilizers and distances to vertices of a given color

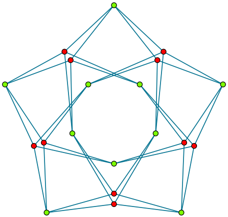


Figure: Folkman’s graph, the smallest semi-symmetric graph. [2]

Problem Statement

- What is the smallest 5-valent semi-symmetric graph?
  - typically proving this is hard, and is done either by enumeration or long combinatorial arguments
  - algorithms to brute-force enumerate edge-transitive graphs are too expensive to get past 30 vertices
  - Conder et al. is an exception, where they use powerful results from Goldschmidt to classify graphs [1].
- How can we construct 5-valent semi-symmetric graphs?
  - are there easy constructions?
  - can we find an infinite family?

Previous Results

- The smallest semi-symmetric graph is Folkman's graph on 20 vertices and 40 edges.
- The smallest 3-valent semi-symmetric graph is the Gray graph on 54 vertices.
- A semi-symmetric graph must have n vertices where n is even and not $2p$ or $2p^2$ for any prime p. [2]

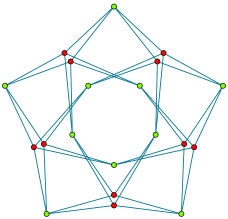


Figure: Folkman's graph, the smallest semi-symmetric graph. [2]

Bi-Coset Construction

Let G be a group and let H, K be subgroups. Construct a graph $\Gamma = \mathrm{bcc}(G; H, K)$ with

$$V(\Gamma) = G/H \cup G/K$$
$$E(\Gamma) = \{(Hg, Kg) | g \in G\}$$

- Hg_1 is adjacent to Kg_2 if and only if $Hg_1 \cap Kg_2 \neq \emptyset$.
- d-valent $\Leftrightarrow [H : H \cap K] = [K : H \cap K] = d$.
- connected $\Leftrightarrow \langle H, K \rangle = G$.
- always edge-transitive, bi-partite (*bi-transitive*).
- all bi-transitive graphs come from this construction: pick $u, v \in V(\Gamma)$ adjacent, let $H = \mathrm{Stab}_G(u), K = \mathrm{Stab}_G(v)$. Then $\mathrm{bcc}(G; H, K) \cong \Gamma$.
- For all $g \in G$, $\mathrm{bcc}(G; H, K) \cong \mathrm{bcc}(G; g^{-1}Hg, g^{-1}Kg)$.

Bi-Coset Construction Searches

How to find d-valent semi-symmetric graphs:

1. Pick a finite group G from a database.
2. For each $H \leq G$ with $d | \# H$, consider a representative K of every conjugacy class of subgroups that can satisfy $[H : H \cap K] = [K : H \cap K] = d$.
3. Compute $\Gamma \cong \mathrm{bcc}(G; H, K)$.
4. Determine if Γ is vertex-transitive.

Searching all finite groups of size less than 1200, I have found three 5-valent semi-symmetric graphs. Only one of these, with 250 vertices, was previously discovered by Lazebnik and Viglione [3].

Question: what are the graphs we have found?

What has been found?

- If Γ is bi-transitive, degree d and $\mathrm{Aut}^+(\Gamma)$ has a subgroup of size $n \leq 1200$ transitive on the edges of Γ then Γ was found by the search.
- In the case where $H \leq \mathrm{Aut}(\Gamma)$ is edge-transitive and $|H| = |E(\Gamma)|$ I have found all semi-symmetric 5-valent graphs with less than 1200 edges.
- When $H \leq \mathrm{Aut}(\Gamma)$ acts on the edges this way, it acts *regularly*. Namely, for $e_1, e_2 \in E(\Gamma)$ there exists exactly one $h \in H$ so that $e_1 h = e_2$. Equivalently, the dart-stabilizers in H are trivial. I call a graph with such an action *edge-regular*.
- Therefore, every edge-regular semi-symmetric graph with less than 1200 edges has been found.

Question: how can I classify which graphs are edge-regular?
Better Question: how can I classify which graphs are not edge-regular?

Cayley Graphs

Let G be a group and $S \subset G$. Define $\Gamma = \text{Cay}(G, S)$ to be the(undirected!) graph with $V(\Gamma) = G$ and $E(\Gamma) = \{\{g, sg\} | g \in G\}$.

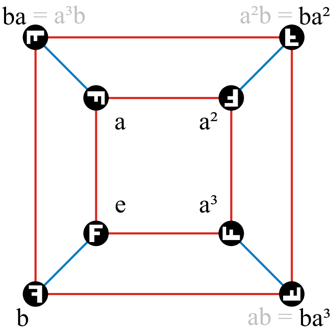
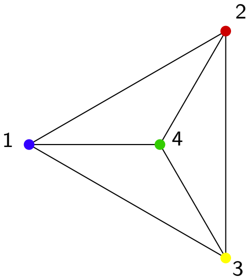


Figure: $\text{Cay}(D_4, \{a, b\})$ where a (red) is rotation and b (blue) is reflection.

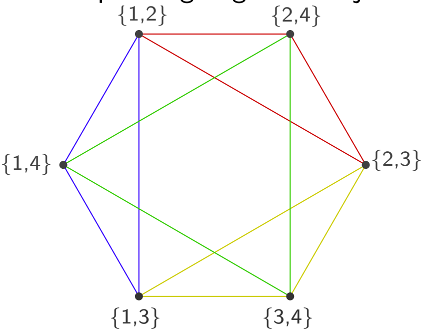
- G acts regularly on the vertices of $\text{Cay}(G, S)$.

Line Graphs

Let Δ be a graph. Define $\Gamma = L(\Delta)$ so that $V(\Gamma) = E(\Delta)$ and two vertices of Γ are adjacent when the corresponding edges are adjacent.



$\mathrm{Aut}(K_4) \cong S_4$



$\mathrm{Aut}(L(K_4)) \cong S_4 \times S_2$

- $L(\Delta)$ usually has more edges than Δ has vertices.
- $\mathrm{Aut}(L(\Delta)) \neq \mathrm{Aut}(\Delta)$ in general.

Line graphs of Edge-Regular Graphs are Cayley Graphs

Motivation: Cayley graphs are vertex-regular, and line graphs "switch" edges and vertices!

Lemma

If G is a group with $H, K \leq G$, $H \cap K = 1$ and $\langle H, K \rangle = G$ then $L(\mathrm{bcc}(G; H, K)) \cong \mathrm{Cay}(G, H \cup K - \{1\})$.

Proof.

Explicitly construct the vertex and edge set of $L(\mathrm{bcc}(G; H, K))$. They match $\mathrm{Cay}(G, H \cup K - \{1\})$ exactly. □

Theorem

A connected bi-transitive graph Δ is edge-regular if and only if there exists a group G and a subset $S \subset G$ such that $L(\Delta) \cong \mathrm{Cay}(G, S)$.

Proof of Theorem

Theorem

A connected bi-transitive graph Δ is edge-regular if and only if there exists a group G and a subset $S \subset G$ such that $L(\Delta) \cong \mathrm{Cay}(G, S)$.

Proof.

($\Rightarrow$). Suppose $G \leq \mathrm{Aut}(\Delta)$ acts regularly on the edges of Δ. Pick H and K to be stabilizers of an adjacent white and black vertex in G, respectively. $H \cap K$ is a dart-stabilizer, so $H \cap K = 1$. $\Delta \cong \mathrm{bcc}(G; H, K)$. The lemma establishes that $L(\Delta) \cong \mathrm{Cay}(G, H \cup K - \{1\})$.

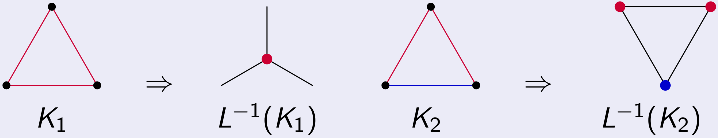
($\Leftarrow$). Outline: Let $\Gamma = L(\Delta)$. For Δ bi-partite, $\mathrm{Aut}(\Gamma)$ acts on $E(\Delta)$ in the same way that $\mathrm{Aut}(\Gamma)$ acts on $V(\Gamma)$ (demonstrated on the next slide). If $\Gamma = \mathrm{Cay}(G, S)$, then $G \leq \mathrm{Aut}(\Gamma)$ acts regularly on the vertices of Γ, and therefore G acts regularly on the edges of Δ. □

Lemma

If Δ *is a bi-partite graph,* $\Gamma = L(\Delta)$ *and* $G \leq \mathrm{Aut}(\Gamma)$*, then* G *acts on* Δ *as a subgroup of* $\mathrm{Aut}(\Delta)$*.*

Proof.

(Sketch) The edges of Γ are colored white and black from the vertices of Δ. Let K be a clique in Γ with $|V(K)| \geq 3$. Suppose let K' be an induced subgraph with 3 vertices. By pigeonhole, two edges must be the same color, say red. Then there are two ways K' could be colored:



$K' = K_2$ is a contradiction; the coloring of Δ must be violated. Therefore, all cliques are of a single color, and maximal ones correspond to a single vertex in Δ. G permutes maximal cliques preserving vertex adjacencies, so G acts on the vertices of Δ preserving edge-adjacency. □

Worthiness

Lemma

For any prime p, every connected, unworthy, bi-partite, edge-transitive graph with valence p is isomorphic to $K_{p.p}$.

Proof.

Suppose $u_1, \ldots, u_s$ is a maximal set of white vertices that have the same neighbours. By edge-transitivity, all white vertices are partitioned into sets of size $s > 1$ that have the same neighbours. If v is black then its p neighbours are partitioned into sets of size s so $s|p \Rightarrow s = p$. The p neighbours of $u_1, \ldots, u_p$ will be black vertices $v_1, \ldots, v_p$. In turn, their neighbours are exactly $u_1, \ldots, u_p$. These form a connected component isomorphic to $K_{p,p}$. □

Corollary

Every 5-valent semi-symmetric graph is worthy.

Summary of Results

- Line graphs of edge-regular bi-transitive graphs are Cayley.
  - I have enumerated these graphs through 1200 edges.
  - This has led to conjectures to generalize Marušič's work [4].
- 5-valent semi-symmetric graphs are worthy.
- A candidate for the smallest 5-valent semi-symmetric graph which is minimal.
- An improved census webpage which now includes several 5-valent bi-transitive graphs.

Next Steps

- Find infinite families of 5-valent semi-symmetric graphs.
  - Generalizing voltage graphs for the 5-valent semi-symmetric graphs found may be useful.
  - It may be possible to generalize some 3-valent families such as Marušič's.
- Develop new search techniques to establish whether the 5-valent semi-symmetric graphs of 120 vertices is indeed minimal.

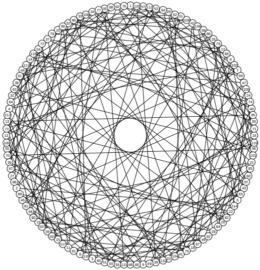


Figure: A 5-valent semi-symmetric graph with 120 vertices.

References

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Questions?

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You can find copies of these slides and a link to the mini-census at
http://www.berkeleychurchill.com/research